Available online at www.sciencedirect.com
ScienceDirect
Journal of Sound and Vibration 316 (2008) 274–297
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

The transition conditions in the dynamics of elastically restrained beams

Ricardo Oscar Grossi*,[1], María Virginia Quintana[1]

Facultad de Ingeniería, Universidad Nacional de Salta, Av. Bolivia 5150, 4400 Salta, Argentina

Received 21 May 2007; received in revised form 26 December 2007; accepted 14 February 2008
Handling Editor: P. Davies
Available online 2 April 2008

Abstract

This paper deals with the free transverse vibration of a non-homogeneous tapered beam subjected to general axial forces, with arbitrarily located internal hinge and elastics supports, and ends elastically restrained against rotation and translation. A rigorous and complete development is presented. First, a brief description of several papers previously published is included. Second, the Hamilton principle is rigorously stated by defining the domain D of the action integral and the space D_a of admissible directions. The differential equations, boundary conditions, and particularly the transitions conditions, are obtained. Third, the transition conditions are analysed for several sets of restraints conditions. Fourth, the existence and uniqueness of the weak solutions of the boundary value problem and the eigenvalue problem which, respectively, govern the statical and dynamical behaviour of the mentioned beam is treated. Finally, the method of separation of variables is used for the determination of the exact frequencies and mode shapes and a modern application of the Ritz method to obtain approximate eigenvalues. In order to obtain an indication of the accuracy of the developed mathematical model, some cases available in the literature have been considered. New results are presented for different boundary conditions and restraint conditions in the internal hinge.

1. Introduction

The calculus of variations is the oldest and most important root of functional analysis. Lagrange invented the ‘‘operator’’ δ and with its application a δ-calculus which was viewed as a kind of ‘‘higher’’ infinitesimal calculus. This discipline has attracted the attention of numerous eminent mathematicians, who made important contributions to its development. In the last decades the interest in application of the techniques of the calculus of variations has increased noticeably. This is partly due to the demands of the technology and the availability of powerful computers.

Variational principles have always played an important role in theoretical mechanics. In 1717 Johann Bernoulli presented the principle of virtual work and in 1835 Hamilton's principle emerged. Particularly, this

*Corresponding author. Tel.: +54 387 4255379; fax: +54 387 4255351.
E-mail address: grossiro@unsa.edu.ar (R.O. Grossi).
[1]A research member of CONICET, INIQUI-ICMASA.

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2008.02.019

Nomenclature

$A_i(x)$	cross-sectional area of the ith span, $i=1,2$
$D(F)$	domain of functional F
$D_a(F)$	space of admissible directions
$D_i(x)=E_i(x)I_i(x)$	flexural rigidity of the ith span
$F(u)$	energy functional
$f(x)$	distributed axial force
K_{ri}, K_{ti}, $i=1,2$	dimensionless rotational and translational parameters
K_{rc}, K_{tc}	dimensionless rotational and translational parameters
K_{r12}	dimensionless rotational parameter
l	length of the beam
$m_i(x)=\rho_i(x)A_i(x)$	mass per unit length of the ith span
$\mathbb{N}$	the set of natural numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{R}$	the set of real numbers
r_1, r_2	rotational stiffness at the left and right ends, respectively
r_c	rotational stiffness at the point $x=c$
r_{12}	rotational stiffness at the internal hinge
S_i	dimensionless axial force
$T(x)$	axial load at abscissa x
T_b	kinetic energy
t	time
t_1, t_2	translational stiffness at the left and right ends, respectively
t_c	translational stiffness at the point $x=c$
U	strain energy
x	abscissa
$\bar{x}$	dimensionless abscissa
$\lambda_{1,i}=\sqrt[4]{\frac{m_1}{D_1}\omega_i^2}\,l$	dimensionless natural frequency parameter
$\delta F(u;v)$	variation of functional F
ω	radian frequency
$\rho_i(x)$	mass density of the ith span

last principle provides a straightforward method for determining equations of motion and boundary conditions of mechanical systems. Substantial literature has been devoted to the theory and applications of the calculus of variations. For instance, the excellent books [1–3] present clear and rigorous treatments of the theoretical aspects of the mentioned discipline. Several classical textbooks, [4–8] present formulations, by means of variational techniques, of boundary value and eigenvalue problems in the statics and dynamics of mechanical systems.

On the other hand, the study of vibration problems of beams with several complicating effects has received considerable treatment. It is not possible to give a detailed account because of the great amount of information, nevertheless some references will be cited. Excellent handbooks have appeared in the literature giving frequency tables and mode-shape expressions [9,10]. Several investigators have studied the influence of rotational and/or translational restraints at the ends of vibrating beams. A number of previous papers have been published on uniform beams with elastically restrained ends [11–18]. Transverse vibrations of beams of non-uniform cross sections have also been extensively investigated [19–27].

Also, the study on vibration of beams with intermediate elastic restraints has been performed by several researchers. Rutemberg [28] presented eigenfrequencies for a uniform cantilever beam with a rotational restraint at an intermediate position. Lau [29] extended Rutemberg's results including an additional spring to against translation. Rao [30] analysed the frequencies of a clamped–clamped uniform beam with intermediate elastic support. De Rosa et al. [31] studied the free vibrations of stepped beams with intermediate elastic supports. Arenas and Grossi [32] presented exact and approximate frequencies of a uniform beam, with one end spring-hinged and a rotational restraint in a variable position. Grossi and Albarracín [33] determined the exact eigenfrequencies of a uniform beam with intermediate elastic constraints. Wang [34] determined the minimum stiffness of an internal elastic support to maximize the fundamental frequency of a vibrating beam.

A review of the literature further reveals that there is only a limited amount of information for the vibration of beams with internal hinges. Ewing and Mirsafian [35] analysed the forced vibrations of two beams joined with a nonlinear rotational joint. Wang and Wang [36] studied the fundamental frequency of a beam with an internal hinge and subjected to an axial force. Chang et al. [37] investigated the dynamic response of a beam with an internal hinge, subjected to a random moving oscillator. The aim of the present paper is to investigate the natural frequencies and mode shapes of a beam with several complicating effects, intending the development within each section to be rigorous and complete.

Modern developments in engineering are making increasing use of several mathematical theories that in the past have been considered as tools of pure mathematicians. A typical method of solving boundary and eigenvalue problems for elliptic partial differential equations with variable coefficients is the variational method.

In most cases of interest in engineering, there exists a variational problem, equivalent to the boundary or eigenvalue problem considered. But the differential equation involves unnecessarily derivatives of higher order than the order of the derivatives included in the corresponding functional that describes certain type of energy. So, it is more natural, from a physical point of view, to look for the weak solution of the given problem than to look for its classical solution [38–46]. Since the restrictions on smoothness for weak solutions are milder than those for classical solutions, the variational approach extends the set of problems that can be investigated. Moreover, the classical solution, does not exist for many important engineering and mathematical physics problems.

One of the reasons of the present paper is to present a rigorous procedure, by formulating the stationary condition for the functional $\int_{t_a}^{t_b}(T_b - U)\,\mathrm{d}t$, involved in Hamilton's principle, in the space of admissible functions D and the space D_a of admissible directions. Another motivation is to determine sufficient conditions for the existence and uniqueness of the weak solution of the boundary value problem and of the eigenvalue problem which, respectively, governs the static and dynamical behaviour of the mechanical system under study.

It is also the purpose of the present paper to determine the natural frequencies and the effects of the elastic restraints of the described system. The method of separation of variables is used for the determination of the exact frequencies and mode shapes. In addition, several cases are solved by the Ritz method with systems of simple polynomials as bases. In order to obtain an indication of the accuracy of the developed mathematical model, some cases available in the literature, have been considered, and comparisons of numerical results are included. The algorithms developed can be applied to a wide range of elastic restraint conditions, different material characteristics, step changes in cross-section and in axial force and distributed axial forces. The generally restrained beam analysed includes the classical end conditions: clamped, simply supported, sliding and free as simply particular cases. The effects of the variations of the elastic restraints at the ends, at the intermediate point and at the internal hinge on the dynamics characteristics are investigated. The transitions conditions are particularly analysed, since these conditions are essential to study the action of an internal hinge with a rotational restraint and the action of intermediate rotational and translational restraints.

Tables and figures are given for frequencies, and in some selected cases, two-dimensional plots for mode shapes are included. A great number of problems were solved and, since this number of cases is prohibitively large, results are presented for only a few cases. The present paper is organized in the following way. First the brief history, stated above. In Section 2, a rigorous treatment of techniques of the calculus of variations to obtain the governing differential equations, the boundary conditions and the transitions conditions is presented. In Section 3 the transitions conditions are analysed. In Section 4 the determination of sufficient conditions for the existence and uniqueness of the weak solutions is included. Finally, in Section 5, the method of separation of variables is used for the determination of the exact frequencies and mode shapes, and a modern application of the Ritz method is used to obtain approximate eigenvalues.

2. Variational derivation of the boundary and eigenvalue problem

Let us consider the tapered beam of length l, which has elastically restrained ends, is constrained at an intermediate point and has an internal hinge elastically restrained against rotation, as shown in Fig. 1. The beam system is made up of two different spans, which correspond to the intervals $[0, c]$ and $[c, l]$, respectively, with variable mass per unit length and variable flexural rigidity of the ith span as $m_i(x) = \rho_i(x)A_i(x)$ and $D_i(x) = E_i(x)I_i(x)$. It is assumed that the ends, the intermediate point c and the hinge are elastically restrained against translation and/or rotation. The rotational restraints are characterised by the spring constants r_1, r_2, r_{12} and r_c, and the translational restraints by the spring constants t_1, t_2 and t_c. Adopting the adequate values of the parameters r_i and t_i, $i = 1, 2$, all the possible combinations of classical end conditions, (i.e.: clamped, pinned, sliding and free) can be generated. On the other hand, adopting the adequate values of the parameters r_c, r_{12} and t_c different constraints on the point $x = c$ and on the hinge can be generated.

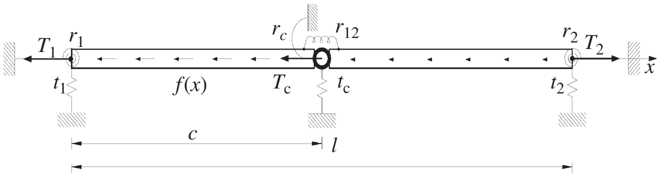


Fig. 1. The elastically restrained beam with an internal hinge and intermediate supports.

It is also assumed that the beam is subjected to an axial force T_1 at the left end, a force T_c at the point c, a force T_2 at the right end, and a distributed force $f(x)$, with values such that the mechanical system is in equilibrium and determine a general axial force $T(x)$, $\forall x \in [0, l]$. Several cases can be analysed, for instance if $T_1 \neq 0$, $T_c \neq 0$, $f(x) = 0$, $\forall x \in [0, l]$, then $T_2 = T_1 + T_c$, $T(x) = T_1(x) = T_1$, $\forall x \in [0, c)$ $T(x) = T_2(x) = T_1 + T_c$, $\forall x \in (c, l]$.

In order to analyse the transverse planar displacements of the system under study, we suppose that the vertical position of the beam at any time t is described by the function $u = u(x, t)$, $x \in [0, l]$. It is well known that at time t, the kinetic energy of the beam can be expressed as

$$T_b = \frac{1}{2}\int_0^c m_1(x)\left(\frac{\partial u(x,t)}{\partial t}\right)^2 \mathrm{d}x + \frac{1}{2}\int_c^l m_2(x)\left(\frac{\partial u(x,t)}{\partial t}\right)^2 \mathrm{d}x. \tag{1}$$

Since the beam is subjected to the axial tensile force $T(x)$, the total potential energy due to the elastic deformation of the beam, the springs at the ends restraints and the springs at the intermediate restraints is given by

$$\begin{aligned} U = \frac{1}{2}\Bigg\{ & \int_0^c D_1(x)\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^2 \mathrm{d}x + \int_c^l D_2(x)\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^2 \mathrm{d}x \\ & + \int_0^c T_1(x)\left(\frac{\partial u(x,t)}{\partial x}\right)^2 \mathrm{d}x + \int_c^l T_2(x)\left(\frac{\partial u(x,t)}{\partial x}\right)^2 \mathrm{d}x \\ & + r_1\left(\frac{\partial u(0^+,t)}{\partial x}\right)^2 + t_1 u^2(0^+,t) + r_c\left(\frac{\partial u(c^-,t)}{\partial x}\right)^2 \\ & + r_{12}\left(\frac{\partial u(c^+,t)}{\partial x} - \frac{\partial u(c^-,t)}{\partial x}\right)^2 + t_c u^2(c,t) + r_2\left(\frac{\partial u(l^-,t)}{\partial x}\right)^2 + t_2 u^2(l^-,t)\Bigg\}, \end{aligned} \tag{2}$$

where the notations 0^+, c^-, c^+ and l^- imply the use of lateral limits and lateral derivatives. It can be observed that the strain energy due to the rotational restraint of coefficient r_c, is computed by means of the expression $r_c/2\left(\partial u(c^-,t)/\partial x\right)^2$, which implies that the spring is connected at right end of the span which corresponds to the interval $[0, c]$, and is connected to a fixed wall. On the other hand, the strain energy, which corresponds to the rotational restraint of the internal hinge, is computed by $r_{12}/2\left(\left(\partial u(c^+,t)/\mathrm{d}x\right) - \left(\partial u(c^-,t)/\mathrm{d}x\right)\right)^2$, which implies that the spring is connected at right end of the first span and at the left end of the second span.

Calculus of variations is a discipline in which the "operator" δ has been assigned special properties not subsumed in the rigorous formalism of mathematics. A mechanical "δ-method" has been developed and extensively used. In the current engineering literature, its use can be observed with heuristics developments. This lack of rigour can arise as a disadvantage, but fortunately, it can be easily overcome, since the variation δI of a functional is a straightforward generalization of the definition of the directional derivative of a real valued function defined on a subset of $\mathbb{R}^n$. In the present paper a rigorous formulation of the Hamilton's principle is presented. The procedure adopted is particularly important in the determination of the analytical expression of the corresponding boundary conditions and transition conditions.

Hamilton's principle requires that between times t_a and t_b, at which the positions are known, the motion will make stationary the action integral $F(u) = \int_{t_a}^{t_b} L\,\mathrm{d}t$ on the space of admissible functions, where the Lagrangian

L is given by $L = T_b - U$, [2]. In consequence, the energy functional to be considered is given by

$$F(u) = \frac{1}{2}\int_{t_a}^{t_b}\left[\sum_{i=1}^{2}\int_{\Omega_i}\left(m(x)\left(\frac{\partial u(x,t)}{\partial t}\right)^2 - D(x)\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^2 - T(x)\left(\frac{\partial u(x,t)}{\partial x}\right)^2\right)\mathrm{d}x\right]\mathrm{d}t - \frac{1}{2}\int_{t_a}^{t_b}\sum_{i=1}^{3} a_i\left(\frac{\partial u(b_i,t)}{\partial x}\right)^2\mathrm{d}t - \frac{1}{2}\int_{t_a}^{t_b} r_{12}\left(\frac{\partial u(d_2,t)}{\partial x} - \frac{\partial u(b_2,t)}{\partial x}\right)^2\mathrm{d}t - \frac{1}{2}\int_{t_a}^{t_b}\sum_{i=1}^{3} c_i u^2(d_i,t)\,\mathrm{d}t, \quad (3)$$

where

$$m(x) = \begin{cases} m_1(x) = \rho_1(x)A_1(x), & \forall x \in [0,c), \\ m_2(x) = \rho_2(x)A_2(x), & \forall x \in (c,l], \end{cases}$$

$$D(x) = \begin{cases} D_1(x) = E_1(x)I_1(x), & \forall x \in [0,c), \\ D_2(x) = E_2(x)I_2(x), & \forall x \in (c,l], \end{cases}$$

$$T(x) = \begin{cases} T_1(x), & \forall x \in [0,c), \\ T_2(x), & \forall x \in (c,l], \end{cases}$$

$\Omega_1 = (0, c)$, $\Omega_2 = (c, l)$, and the coefficients a_i, b_i, c_i and d_i are defined in Table 1. The stationary condition for the functional given by Eq. (3) requires that

$$\delta F(u,v) = 0, \quad \forall v \in D_a, \quad (4)$$

where $\delta F(u, v)$ is the first variation of F at u in the direction v and D_a is the space of admissible directions at u for the space D of admissible functions. In order to make the mathematical developments required by the application of the techniques of the calculus of variations, we assume that $m_i(x) \in C(\bar{\Omega}_i)$, $D_i(x) \in C^2(\bar{\Omega}_i)$, $T_i(x) \in C^1(\bar{\Omega}_i)$, $i = 1,2$ with $\bar{\Omega}_1 = \Omega_1 \cup \{0,c\}$ and $\bar{\Omega}_2 = \Omega_2 \cup \{c,l\}$. The space D is the set of functions $u(x,\bullet) \in C^2[t_a,t_b]$, $u(\bullet,t) \in C(\bar{\Omega})$, $u(\bullet,t)\big|_{\bar{\Omega}_i} \in C^4(\bar{\Omega}_i)$, $i = 1,2$, where $\Omega = (0,l)$ and $\bar{\Omega} = \Omega \cup \{0,l\}$. It must be noted that the classical derivatives $(\partial^n u(x,t))/\partial x^n$, with $n = 2,3,4$ do not necessarily exist in the interval Ω so it is necessary to impose the conditions $u(\bullet,t)\big|_{\bar{\Omega}_i} \in C^4(\bar{\Omega}_i)$, $i = 1,2$. Since $(\partial^n u(x,t))/\partial x^n$, $n = 2,3,4$ are continuously extended to the extreme $x = c$, the lateral derivatives $(\partial^n u(c^-,t))/\partial x^n$ and $(\partial^n u(c^+,t))/\partial x^n$ both exist, but $(\partial^n u(x,t))/\partial x^n$ is not necessarily continuous at $x = c$, nevertheless it has at most a discontinuity of the first kind at this point.

In view of all these observations and since Hamilton's principle requires that at times t_a and t_b the positions are known, the space D is given by

$$D = \left\{u; u(x,\bullet) \in C^2[t_a,t_b], u(\bullet,t) \in C(\bar{\Omega}), u(\bullet,t)\big|_{\bar{\Omega}_i} \in C^4(\bar{\Omega}_i), i = 1,2, \quad u(x,t_a), u(x,t_b) \text{ prescribed}\right\}. \quad (5)$$

Table 1
Definition of coefficients a_i, b_i, c_i and d_i in Eq. (3)

i	a_i	b_i	c_i	d$_i$
1	r_1	0^+	t_1	0^+
2	r_c	c^-	t_c	c^+
3	r_2	l^-	t_2	l^-

The only admissible directions v at $u \in D$ are those for which $u + \varepsilon v \in D$ for sufficiently small ε and $\delta F(u; v)$ exists. In consequence, and in view of Eq. (5), v is an admissible direction at u for D if, and only if, $v \in D_a$ where

$$D_a = \left\{ v; v(x, \bullet) \in C^2[t_a, t_b], v(\bullet, t) \in C(\bar{\Omega}), v(\bullet, t)\big|_{\bar{\Omega}_i} \in C^4(\bar{\Omega}_i), i = 1, 2, \quad v(x, t_a) = v(x, t_b) = 0, \forall x \in [0, l] \right\}. \tag{6}$$

Now it is possible to introduce the definition of the variation of F at u in the direction v, as a generalization of the definition of the directional derivative of a real valued function defined on a subset of $\mathbb{R}^n$, [2]. Consequently, the definition of the first variation of F at u in the direction v, is given by

$$\delta F(u; v) = \left.\frac{\mathrm{d}F(u + \varepsilon v)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}. \tag{7}$$

The application of Eq. (7) leads to

$$\begin{aligned}\delta F(u; v) = &\int_{t_a}^{t_b} \left[\sum_{i=1}^{2} \int_{\Omega_i} \left(m(x) \frac{\partial u(x, t)}{\partial t} \frac{\partial v(x, t)}{\partial t} - D(x) \frac{\partial^2 u(x, t)}{\partial x^2} \frac{\partial^2 v(x, t)}{\partial x^2} \right.\right. \\ &\left.\left. - T(x) \frac{\partial u(x, t)}{\partial x} \frac{\partial v(x, t)}{\partial x} \right) \mathrm{d}x \right] \mathrm{d}t - \int_{t_a}^{t_b} \sum_{i=1}^{3} a_i \frac{\partial u(b_i, t)}{\partial x} \frac{\partial v(b_i, t)}{\partial x} \, \mathrm{d}t \\ &- \int_{t_a}^{t_b} r_{12} \left(\frac{\partial u(d_2, t)}{\partial x} - \frac{\partial u(b_2, t)}{\partial x} \right) \left(\frac{\partial v(d_2, t)}{\partial x} - \frac{\partial v(b_2, t)}{\partial x} \right) \mathrm{d}t - \int_{t_a}^{t_b} \sum_{i=1}^{3} c_i u(d_i, t) v(d_i, t) \, \mathrm{d}t. \end{aligned} \tag{8}$$

A procedure of integration by parts, transforms Eq. (8) in

$$\begin{aligned}\delta F(u; v) = &- \int_{t_a}^{t_b} \left\{ \sum_{i=1}^{2} \int_{\Omega_i} \left[m(x) \frac{\partial^2 u(x, t)}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left(D(x) \frac{\partial^2 u(x, t)}{\partial x^2} \right) \right.\right. \\ &\left.\left. - \frac{\partial}{\partial x} \left(T(x) \frac{\partial u(x, t)}{\partial x} \right) \right] v(x, t) \, \mathrm{d}x \right\} \mathrm{d}t + \int_{t_a}^{t_b} \sum_{i=1}^{2} \left(P_i \frac{\partial v(A_i, t)}{\partial x} + Q_i v(A_i, t) \right. \\ &\left. + R_i \frac{\partial v(B_i, t)}{\partial x} + Z_i v(c, t) \right) \mathrm{d}t, \end{aligned} \tag{9}$$

where $A_1 = 0^+, \; A_2 = l^-, \; B_1 = c^-, \; B_2 = c^+$. The algebraic details to obtain Eq. (9) and the analytical expressions of the terms P_i, Q_i, R_i and Z_i are given in Appendix A.

Now it is convenient to consider the directions $v(x, t)$ which satisfy

$$v(c, t) = v(A_i, t) = \frac{\partial v(A_i, t)}{\partial x} = \frac{\partial v(B_i, t)}{\partial x} = 0, \quad i = 1, 2, \quad \forall t \in (t_a, t_b). \tag{10a–g}$$

Substituting Eqs. (10a–g) into Eq. (9) and applying the stationary condition required by Hamilton's principle given by Eq. (4), leads to

$$\begin{aligned}\delta F(u; v) = &\int_{t_a}^{t_b} \sum_{i=1}^{2} \int_{\Omega_i} \left[m(x) \frac{\partial^2 u(x, t)}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left(D(x) \frac{\partial^2 u(x, t)}{\partial x^2} \right) \right. \\ &\left. - \frac{\partial}{\partial x} \left(T(x) \frac{\partial u(x, t)}{\partial x} \right) \right] v(x, t) \, \mathrm{d}x \, \mathrm{d}t = 0, \quad \forall v \in D_a. \end{aligned} \tag{11}$$

Let us assume $t_a = 0$, then as $v(x,t)$ is an arbitrary smooth function, the fundamental lemma of the calculus of variations can be applied to Eq. (11) to conclude that the function $u(x, t)$ must satisfy the following differential equations:

$$\begin{aligned}&\frac{\partial^2}{\partial x^2} \left(D(x) \frac{\partial^2 u(x, t)}{\partial x^2} \right) - \frac{\partial}{\partial x} \left(T(x) \frac{\partial u(x, t)}{\partial x} \right) + m(x) \frac{\partial^2 u(x, t)}{\partial t^2} = 0, \\ &\forall x \in \Omega_i, \quad i = 1, 2, \quad t \geqslant 0. \end{aligned} \tag{12a,b}$$

Now it is possible to remove the restrictions given by Eqs. (10a–g), and since the function $u(x,t)$ must satisfy the differential equations stated above, Eq. (9) is reduced to

$$\delta F(u; v) = -\int_0^{t_b} \sum_{i=1}^{2} \left(P_i \frac{\partial v(A_i, t)}{\partial x} + Q_i v(A_i, t) + R_i \frac{\partial v(B_i, t)}{\partial x} + Z_i v(c, t) \right) \mathrm{d}t. \tag{13}$$

Since, the functions $(\partial v(A_i, t))/\partial x$, $v(A_i, t)$, $(\partial v(B_i, t))/\partial x$ and $v(c, t)$ are smooth and arbitrary, the stationary condition given by Eq. (4) applied to Eq. (13), leads to the boundary and transitions conditions. For instance, if we adopt

$$v(0^+, t) = \frac{\partial v(c^-, t)}{\partial x} = \frac{\partial v(c^+, t)}{\partial x} = v(c, t) = \frac{\partial v(l^-, t)}{\partial x} = v(l^-, t) = 0, \quad \forall t \in (0, t_b) \tag{14a–f}$$

we obtain

$$r_1 \frac{\partial u(0^+, t)}{\partial x} = D(0^+) \frac{\partial^2 u(0^+, t)}{\partial x^2}. \tag{15}$$

In an analogue form all the rest of the boundary conditions and transitions conditions are obtained. It has been demonstrated that the function $u(x, t)$ must satisfy the boundary and eigenvalue problem shown in Table 2.

3. The transition conditions

Since the domain of definition of the problem is $\Omega = (0, l)$ and this is an open interval in $\mathbb{R}$, the boundary is given by two points, i.e. $\partial\Omega = \{0, l\}$. Consequently, only Eqs. (18), (19), (23) and (24) correspond to the boundary conditions. The point $x = c$ is an interior point of Ω, and the equations formulated at $x = c^-$ and $x = c^+$ can be called transition conditions. Consequently, Eqs. (20)–(22) correspond to the transition

Table 2
Boundary and eigenvalue problem

$$\frac{\partial^2}{\partial x^2}\left(D(x)\frac{\partial^2 u(x,t)}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(T(x)\frac{\partial u(x,t)}{\partial x}\right) + m(x)\frac{\partial^2 u(x,t)}{\partial t^2} = 0, \quad \forall x \in (0, c), \tag{16}$$

$$\frac{\partial^2}{\partial x^2}\left(D(x)\frac{\partial^2 u(x,t)}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(T(x)\frac{\partial u(x,t)}{\partial x}\right) + m(x)\frac{\partial^2 u(x,t)}{\partial t^2} = 0, \quad \forall x \in (c, l), \tag{17}$$

$$r_1 \frac{\partial u(0^+, t)}{\partial x} = D(0^+)\frac{\partial^2 u(0^+, t)}{\partial x^2}, \tag{18}$$

$$t_1 u(0^+, t) = -\frac{\partial}{\partial x}\left(D(0^+)\frac{\partial^2 u(0^+, t)}{\partial x^2}\right) + T(0^+)\frac{\partial u(0^+, t)}{\partial x}, \tag{19}$$

$$r_{12}\left(\frac{\partial u(c^+, t)}{\partial x} - \frac{\partial u(c^-, t)}{\partial x}\right) = D(c^+)\frac{\partial^2 u(c^+, t)}{\partial x^2}, \tag{20}$$

$$r_{12}\left(\frac{\partial u(c^+, t)}{\partial x} - \frac{\partial u(c^-, t)}{\partial x}\right) - r_c\frac{\partial u(c^-, t)}{\partial x} = D(c^-)\frac{\partial^2 u(c^-, t)}{\partial x^2}, \tag{21}$$

$$t_c u(c, t) = \frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-, t)}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+, t)}{\partial x^2}\right) - T(c^-)\frac{\partial u(c^-, t)}{\partial x} + T(c^+)\frac{\partial u(c^+, t)}{\partial x}, \tag{22}$$

$$r_2 \frac{\partial u(l^-, t)}{\partial x} = -D(l^-)\frac{\partial^2 u(l^-, t)}{\partial x^2}, \tag{23}$$

$$t_2 u(l^-, t) = \frac{\partial}{\partial x}\left(D(l^-)\frac{\partial^2 u(l^-, t)}{\partial x^2}\right) - T(l^-)\frac{\partial u(l^-, t)}{\partial x}, \quad \text{where } t \geqslant 0. \tag{24}$$

conditions of the problem. Since $u(\bullet, t) \in C(\bar{\Omega})$, there exists continuity of deflection at the point $x = c$ and this generate the transition condition $u(c^-, t) = u(c^+, t) = u(c, t)$. If Eq. (21) is subtracted from Eq. (20) we have

$$r_c \frac{\partial u(c^-, t)}{\partial x} = D(c^+)\frac{\partial^2 u(c^+, t)}{\partial x^2} - D(c^-)\frac{\partial^2 u(c^-, t)}{\partial x^2}. \tag{25}$$

In consequence, the set of all the transitions conditions of the problem can be expressed as

$$u(c^-, t) = u(c^+, t), \tag{26}$$

$$r_c \frac{\partial u(c^-, t)}{\partial x} = D(c^+)\frac{\partial^2 u(c^+, t)}{\partial x^2} - D(c^-)\frac{\partial^2 u(c^-, t)}{\partial x^2}, \tag{27}$$

$$r_{12}\left(\frac{\partial u(c^+, t)}{\partial x} - \frac{\partial u(c^-, t)}{\partial x}\right) = D(c^+)\frac{\partial^2 u(c^+, t)}{\partial x^2}, \tag{28}$$

$$t_c u(c, t) = \frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-, t)}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+, t)}{\partial x^2}\right) - T(c^-)\frac{\partial u(c^-, t)}{\partial x} + T(c^+)\frac{\partial u(c^+, t)}{\partial x}, \quad t \geqslant 0. \tag{29}$$

It is well known that for a differential equation of order $2m$, the boundary conditions containing the function u and derivatives of u of orders not greater than $m-1$, are called *stable* or *geometric*, and those containing derivatives of orders higher than $m-1$, are called *unstable* or *natural,* [47,48]. In consequence, if $0 \leqslant r_i < \infty$, $0 \leqslant t_i < \infty$, $i = 1,2$, the boundary conditions given by Eqs. (18), (19), (23) and (24) are all unstable. If this classification is extended to the transition conditions, we conclude that if $0 \leqslant r_c < \infty$, $0 \leqslant r_{12} < \infty$, $0 \leqslant t_c < \infty$ the transition conditions given by Eqs. (27)–(29) are unstable. This classification is particularly important when using the Ritz method since we must choose a sequence of functions v_i which constitutes a base in the space of homogeneous stable boundary conditions, as it is demonstrated in the next point. So, in this case, there is no need to subject the functions v_i to the natural boundary and transition conditions. Different situations can be generated by substituting values and/or limiting values of the restraint parameters r_c, r_{12} and t_c into Eqs. (27)–(29). Let us consider $r_{12} = \infty$, $r_c = 0$, $t_c = 0$, $T(x) = 0$, $\forall x \in [0, l]$. The transition conditions given by Eqs. (26)–(29) are reduced to

$$u(c^-, t) = u(c^+, t), \tag{30}$$

$$\frac{\partial u(c^-, t)}{\partial x} = \frac{\partial u(c^+, t)}{\partial x}, \tag{31}$$

$$D(c^-)\frac{\partial^2 u(c^-, t)}{\partial x^2} = D(c^+)\frac{\partial^2 u(c^+, t)}{\partial x^2}, \tag{32}$$

$$\frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-, t)}{\partial x^2}\right) = \frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+, t)}{\partial x^2}\right). \tag{33}$$

There is no internal hinge and the articulation is perfectly rigid. Now let us consider $r_{12} = 0$, $r_c = 0$, $t_c = 0$, $T(x) = 0$, $\forall x \in [0, l]$. The transition conditions given by Eqs. (26)–(29) are reduced to

$$u(c^-, t) = u(c^+, t), \tag{34}$$

$$D(c^-)\frac{\partial^2 u(c^-, t)}{\partial x^2} = D(c^+)\frac{\partial^2 u(c^+, t)}{\partial x^2} = 0, \tag{35a,b}$$

$$\frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-, t)}{\partial x^2}\right) = \frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+, t)}{\partial x^2}\right). \tag{36}$$

There is an internal hinge and the articulation is perfect. Finally if we have $0<r_{12}<\infty$, $0<t_c<\infty$, $r_c=0$, $T=T(x)$, $\forall x\in[0,\, l]$, the transition conditions given by Eqs. (26)–(29) are reduced to

$$u(c^-,t)=u(c^+,t), \tag{37}$$

$$D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}=D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}, \tag{38}$$

$$r_{12}\left(\frac{\partial u(c^+,t)}{\partial x}-\frac{\partial u(c^-,t)}{\partial x}\right)=D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}, \tag{39}$$

$$t_c u(c,t)=\frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}\right)-T(c^-)\frac{\partial u(c^-,t)}{\partial x}+T(c^+)\frac{\partial u(c^+,t)}{\partial x}. \tag{40}$$

There is an internal hinge elastically restrained against rotation and supported by a translational restraint.

Following this procedure, all the transition conditions for the most relevant cases can be obtained. These are shown in Table A1 of Appendix A. The boundary conditions are not included since they are presented in several textbooks, see for instance [9,10].

4. The weak solution

Variational methods were extensively used by engineers and scientists as a very effective tool for the solution of boundary and/or eigenvalue problems. At the same time several problems emerged of both theoretical and practical character. The finite element method is the most widely used technique for engineering design and analysis. This method provides a formalism for generating finite algorithms for approximating the solutions of boundary and/or eigenvalue problems. It works as a black box in which one puts the boundary and/or eigenvalue problem data and out of which is generated an algorithm for approximating the corresponding solutions. A part of this task can be done automatically by a computer, but an amount of mathematical skill is necessary. It proved that to find an answer to a number of questions which are theoretically interesting and practically urgent is not a simple task. The functional analysis plays an essential role in the solutions of these problems and particularly the theory of Sobolev spaces and the concept of weak solution.

Consider the boundary value problem determined by the differential equation of order $2m$,

$$Au=f \quad \text{on a domain } \Omega \tag{41}$$

and by the boundary conditions

$$B(u)=0 \quad \text{on } \partial\Omega, \tag{42}$$

where $\partial\Omega$, denotes the boundary of Ω.

The function u defined for $\bar{\Omega}=\Omega\cup\partial\Omega$ is the classical solution of the boundary value problem (41)–(42) if

(i) $u\in C^{2m}(\Omega)\cap C^{s}(\bar{\Omega})$, where s is the maximum order which appears in Eq. (42).
(ii) Eq. (41) is satisfied for all $x\in\Omega$.
(iii) boundary conditions (42) are satisfied for all $x\in\partial\Omega$.

Condition (i) implies that the classical solution u has continuous derivatives in Ω of those orders which appear in the differential equation (41), and continuous derivatives in $\bar{\Omega}=\Omega\cup\partial\Omega$ of those orders which appear in the boundary conditions (42).

If the function f on the right-hand side of Eq. (41) is discontinuous then Eq. (41) has no classical solution because the derivatives of u of order $2m$ or less, are not continuous in Ω. In this case it is necessary to introduce the concept of weak solution. This situation is naturally extended to eigenvalue problems.

The problem treated in this paper is a typical case of non-existence of classical solution since the presence of the internal hinge implies that the first derivative of u is not continuous in Ω. Another problem is the case of a vibrating beam elastically restrained against rotation in an intermediate point. The corresponding problems in plates lead also to boundary value problems which do not have classical solutions.

4.1. The statical case

The classical solution of the boundary and eigenvalue problem presented in Table 2, when there is no restrictions or hinge at the intermediate point c is a function $u(x, t)$ such that $u(x,\bullet) \in C^2[t_a, t_b]$, $u(\bullet, t) \in C^4(0, l)$ and $u(\bullet, t) \in C^2[0, l]$. In other words, $u(\bullet, t)$ must have fourth-order partial derivatives continuous in the open interval $\Omega = (0, l)$, (since it must satisfy the differential equation) and two-order derivatives and function values continuous in the close interval $\bar{\Omega} = [0, l]$ since it must satisfy the boundary conditions which involve the extremes of this interval. But when there exist an internal hinge and elastic restraints in c, as it has been shown in Section 2, the function $u(x, t)$ does not have derivatives $(\partial^n u(x,t))/\partial x^n$, with $n \geqslant 2$ in the interval Ω. In consequence, the boundary and eigenvalue problem presented in Table 2, does not have a classical solution. So, it is necessary to analyse the existence of a weak solution. Let us consider the statical behaviour of the mechanical system described, when a load $q = q(x)$, which causes a transverse deflection $w(x)$, is applied. It is governed by the corresponding boundary value problem presented in Table 3, which was obtained with an analogue procedure to that used in Section 2. Let $H^2(\Omega)$ be the Sobolev space $H^2(\Omega) = \{u \in L^2(\Omega);\ D^\alpha u \in L^2(\Omega),\ \alpha = 1, 2\}$, where $\Omega = (0,\ l)$. This space can be equipped with the norm $\|u\|_{H^2(\Omega)} = \left(\sum_{\alpha=0}^{2} \int_\Omega (D^\alpha u)^2\, \mathrm{d}x\right)^{1/2}$, where $D^\alpha u = u^{(\alpha)}$ is the weak derivative of order α of the function u.

The stable and unstable boundary and transition conditions are of different nature so in order to clearly distinguish them, it is useful to introduce the space V of elements of the Sobolev space $H^2(\Omega)$, which satisfy the corresponding stable homogeneous boundary and transition conditions. For instance, if we let r_1, r_c, r_{12}, t_1, $t_c \to \infty$, in Eqs. (45)–(49), these conditions are reduced to $w(0^+) = \mathrm{d}w(0^+)/\mathrm{d}x = w(c) = \mathrm{d}w(c^-)/$

Table 3
Boundary value problem $\Omega_1 = (0, c)$, $\Omega_2 = (c, l)$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(D(x)\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(T(x)\frac{\mathrm{d}w(x)}{\mathrm{d}x}\right) = q_1(x), \quad \forall x \in \Omega_1, \tag{43}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(D(x)\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(T(x)\frac{\mathrm{d}w(x)}{\mathrm{d}x}\right) = q_2(x), \quad \forall x \in \Omega_2, \tag{44}$$

$$r_1 \frac{\mathrm{d}w(0^+)}{\mathrm{d}x} = D(0^+)\frac{\mathrm{d}^2 w(0^+)}{\mathrm{d}x^2}, \tag{45}$$

$$t_1 w(0^+) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(D(0^+)\frac{\mathrm{d}^2 w(0^+)}{\mathrm{d}x^2}\right) + T(0^+)\frac{\mathrm{d}w(0^+)}{\mathrm{d}x}, \tag{46}$$

$$r_{12}\left(\frac{\mathrm{d}w(c^+)}{\mathrm{d}x} - \frac{\mathrm{d}w(c^-)}{\mathrm{d}x}\right) = D(c^+)\frac{\mathrm{d}^2 w(c^+)}{\mathrm{d}x^2}, \tag{47}$$

$$r_{12}\left(\frac{\mathrm{d}w(c^+)}{\mathrm{d}x} - \frac{\mathrm{d}w(c^-)}{\mathrm{d}x}\right) - r_c\frac{\mathrm{d}w(c^-)}{\mathrm{d}x} = D(c^-)\frac{\mathrm{d}^2 w(c^-)}{\mathrm{d}x^2}, \tag{48}$$

$$t_c w(c) = \frac{\mathrm{d}}{\mathrm{d}x}\left(D(c^-)\frac{\mathrm{d}^2 w(c^-)}{\mathrm{d}x^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(D(c^+)\frac{\mathrm{d}^2 w(c^+)}{\mathrm{d}x^2}\right) - T(c^-)\frac{\mathrm{d}w(c^-)}{\mathrm{d}x} + T(c^+)\frac{\mathrm{d}w(c^+)}{\mathrm{d}x}, \tag{49}$$

$$r_2 \frac{\mathrm{d}w(l^-)}{\mathrm{d}x} = -D(l^-)\frac{\mathrm{d}^2 w(l^-)}{\mathrm{d}x^2}, \tag{50}$$

$$t_2 w(l^-) = \frac{\mathrm{d}}{\mathrm{d}x}\left(D(l^-)\frac{\mathrm{d}^2 w(l^-)}{\mathrm{d}x^2}\right) - T(l^-)\frac{\mathrm{d}w(l^-)}{\mathrm{d}x}. \tag{51}$$

$\mathrm{d}x = \mathrm{d}w(c^+)/\mathrm{d}x = 0$. Consequently, since a weak solution of the boundary value problem defined by Eqs. (43), (45)–(49) is a function from the Sobolev space $H^2(\Omega_1)$, the space V_1 is given by

$$V_1 = \left\{ v_1; v_1 \in H^2(\Omega_1), v_1(0^+) = \frac{\mathrm{d}v_1(0^+)}{\mathrm{d}x} = v_1(c) = \frac{\mathrm{d}v_1(c^-)}{\mathrm{d}x} = 0 \right\}. \tag{52}$$

Similarly, adopting r_2, r_c, r_{12}, t_c, $t_2 \to \infty$, we conclude that a weak solution of boundary value problem defined by Eqs. (44), (47)–(51) is a function from the Sobolev space $H^2(\Omega_2)$, and the space V_2 is given by

$$V_2 = \left\{ v_2; v_2 \in H^2(\Omega_2), v_2(c) = \frac{\mathrm{d}v_2(c^+)}{\mathrm{d}x} = v_2(l^-) = \frac{\mathrm{d}v_2(l^-)}{\mathrm{d}x} = 0 \right\}. \tag{53}$$

If r_c, r_{12}, t_c take finite values, the transition conditions at the point $x = c$ are unstable, so they do not belong to the spaces V_i. Moreover, when also the coefficients r_1, r_2, t_1, t_2 take finite values, there are no stable boundary conditions and the spaces V_i can be taken as $V_i = \{v_i; v_i \in H^2(\Omega_i)\}, \quad i = 1, 2$.

Let $q_i(x) \in C(\bar{\Omega}_i), \quad D_i(x) \in C^2(\bar{\Omega}_i), \quad T_i(x) \in C^1(\bar{\Omega}_i)$ and $w_i = w(\bullet, t)\big|_{\bar{\Omega}_i} \in C^4(\bar{\Omega}_i)$, be the classical solutions for the problem given by Eqs. (43)–(51). Now this boundary value problem is transformed into one that leads to the concept of the weak solution. If we take arbitrary functions $v_i \in V_i$, and multiply Eqs. (43) and (44) respectively, by these functions and integrate each result over the corresponding domain we get

$$\int_{\Omega_i} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(D(x)\frac{\mathrm{d}^2 w_i(x)}{\mathrm{d}x^2}\right) v_i(x)\,\mathrm{d}x - \int_{\Omega_i} \frac{\mathrm{d}}{\mathrm{d}x}\left(T(x)\frac{\mathrm{d}w_i(x)}{\mathrm{d}x}\right) v_i(x)\,\mathrm{d}x = \int_{\Omega_i} q_i(x) v_i(x)\,\mathrm{d}x, \quad i = 1, 2. \tag{54a,b}$$

Integrating by parts the two first integrals we obtain

$$\begin{aligned} &\int_{\Omega_i} D(x)\frac{\mathrm{d}^2 w_i(x)}{\mathrm{d}x^2}\frac{\mathrm{d}^2 v_i(x)}{\mathrm{d}x^2}\,\mathrm{d}x + \int_{\Omega_i} T(x)\frac{\mathrm{d}w_i(x)}{\mathrm{d}x}\frac{\mathrm{d}v_i(x)}{\mathrm{d}x}\,\mathrm{d}x \\ &+ \frac{\mathrm{d}}{\mathrm{d}x}\left(D(x)\frac{\mathrm{d}^2 w_i(x)}{\mathrm{d}x^2}\right) v_i(x)\Big|_{\partial\Omega_i} - D(x)\frac{\mathrm{d}^2 w_i(x)}{\mathrm{d}x^2}\frac{\mathrm{d}v_i(x)}{\mathrm{d}x}\Big|_{\partial\Omega_i} \\ &- T(x)\frac{\mathrm{d}w_i(x)}{\mathrm{d}x} v_i(x)\Big|_{\partial\Omega_i} = \int_{\Omega_i} q_i(x) v_i(x)\,\mathrm{d}x, \quad \forall v_i \in V_i, \quad i = 1, 2. \end{aligned} \tag{55a,b}$$

Summing Eqs. (55a) and (55b) and taking into account the boundary and transition conditions given by Eqs. (45)–(51), we obtain

$$\begin{aligned} B(w, v) = &\int_{\Omega_1} D(x)\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\frac{\mathrm{d}^2 v(x)}{\mathrm{d}x^2}\,\mathrm{d}x + \int_{\Omega_2} D(x)\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\frac{\mathrm{d}^2 v(x)}{\mathrm{d}x^2}\,\mathrm{d}x \\ &+ \int_{\Omega} T(x)\frac{\mathrm{d}w(x)}{\mathrm{d}x}\frac{\mathrm{d}v(x)}{\mathrm{d}x}\,\mathrm{d}x + \sum_{i=1}^{3}\left(a_i\frac{\mathrm{d}w(b_i)}{\mathrm{d}x}\frac{\mathrm{d}v(b_i)}{\mathrm{d}x} + c_i w(d_i) v(d_i)\right) \\ &+ r_{12}\left(\frac{\mathrm{d}w(d_2)}{\mathrm{d}x} - \frac{\mathrm{d}w(b_2)}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}v(d_2)}{\mathrm{d}x} - \frac{\mathrm{d}v(b_2)}{\mathrm{d}x}\right) = \int_{\Omega} q(x) v(x)\,\mathrm{d}x, \quad \forall v \in V, \end{aligned} \tag{56}$$

where

$$w(\bullet, t)\big|_{\Omega_i} = w_i(x), \quad i = 1, 2, \quad q(x) = \begin{cases} q_1(x), & \forall x \in [0, c), \\ q_2(x), & \forall x \in (c, l]. \end{cases}$$

The coefficients a_i, b_i, c_i, d_i are the same defined in Table 1. Finally, the space V is given by $V = \{v; v \in H^1(\Omega), \quad v|_{\Omega_i} \in H^2(\Omega_i), \quad i = 1, 2\}$. It must be noted that if a function $u(x) \in H^2(\Omega)$ it implies that $u(x)$ and $u'(x)$ are continuous in $\Omega \subset \mathbb{R}$. But as stated above, the presence of the internal hinge implies $\mathrm{d}w(c^+)/\mathrm{d}x \neq \mathrm{d}w(c^-)/\mathrm{d}x$. In consequence $w \in H^1(\Omega)$. More precisely $w \in H^1(\Omega)$ and $w|_{\Omega_i} \in H^2(\Omega_i), \quad i = 1, 2$. This is the reason for adopting the space V defined above. The first three terms on the left-hand side of Eq. (56) constitute the bilinear form $A(w, v)$ associated with the differential equations (43) and (44). The other terms which are related to the boundary and transition conditions, correspond to the bilinear form $a(w, v)$. Eq. (56)

now assumes the form

$$B(w,v) = A(w,v) + a(w,v) = \int_{\Omega} qv\,\mathrm{d}x = (q,v)_{L^2(\Omega)}, \quad \forall v \in V. \tag{57}$$

Now we are going to weaken the assumptions. Let $q(x) \in L^2(\Omega)$, $D(x) \in L^{\infty}(\Omega)$ and $T(x) \in L^{\infty}(\Omega)$. A function w is called a weak solution of the boundary value problem defined by Eqs. (43)–(51) if

$$\text{(i)} \quad w \in V, \tag{58}$$

$$\text{(ii)} \quad B(w,v) = (q,v)_{L^2(\Omega)}, \quad \forall v \in V. \tag{59}$$

It must be noted that in Eq. (56) $\mathrm{d}w(x)/\mathrm{d}x$ denotes the first-order weak derivative of the function w in $\Omega = (0, l)$ but $\mathrm{d}^2w(x)/\mathrm{d}x^2$ is not necessarily a weak derivative, since a step function is not in the space $H^1(0, l)$. For this reason the expressions

$$\int_{\Omega_i} D(x) \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} \frac{\mathrm{d}^2 v(x)}{\mathrm{d}x^2}\,\mathrm{d}x, \quad i = 1,2$$

are used, instead of the integral over Ω. If the bilinear form $B(w, v)$ is continuous in V and V-elliptic, the problem under consideration has exactly one weak solution w [47,48]. If we replace $w = v$, into the expression of $B(w, v)$ we can write

$$\begin{aligned} B(w,w) - 2(q,w)_{L^2(\Omega)} = {} & \int_{\Omega_1} D(x)\left(\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\right)^2 \mathrm{d}x + \int_{\Omega_2} D(x)\left(\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \\ & + \int_{\Omega} T(x)\left(\frac{\mathrm{d}w(x)}{\mathrm{d}x}\right)^2 \mathrm{d}x + \sum_{i=1}^{3}\left(a_i\left(\frac{\mathrm{d}w(b_i)}{\mathrm{d}x}\right)^2 + c_i w^2(d_i)\right) \\ & + r_{12}\left(\frac{\mathrm{d}w(c^+)}{\mathrm{d}x} - \frac{\mathrm{d}w(c^-)}{\mathrm{d}x}\right)^2 - 2\int_{\Omega} q(x)w(x)\,\mathrm{d}x. \end{aligned} \tag{60}$$

We can recognize that Eq. (60) is proportional to the potential energy of the system under study. Since the bilinear form $B(w, v)$ is also symmetric, the function $w(x)$ is the weak solution of the problem defined by Eqs. (43)–(51), if and only if it minimizes, in the space V, the functional, [47,48]

$$I(w) = \tfrac{1}{2}B(w,w) - (q,w)_{L^2(\Omega)}, \quad \forall v \in V. \tag{61}$$

The Ritz method can be applied adopting the approximating function

$$w_N(x) = \sum_{i=1}^{N} c_{Ni}\varphi_i(x), \tag{62}$$

where $\varphi_i(x)$ are elements of a base in V. The coefficients c_{Ni} are determined by the condition $I(w_N) = \min$. This procedure leads to the following system of linear equations:

$$\sum_{j=1}^{N} c_{Nj} B(\varphi_i, \varphi_j) = (\varphi_i, q)_{L^2(\Omega)}, \quad i = 1,2,\ldots,N. \tag{63}$$

4.2. The eigenvalue problem

In the case of normal modes of vibrations we take $u(x, t) = w(x)\cos\omega t$, where ω is the natural radian frequency. Consequently Eqs. (16)–(24), are reduced to

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(D(x)\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(T(x)\frac{\mathrm{d}w(x)}{\mathrm{d}x}\right) - \omega^2 m(x)w(x) = 0, \quad \forall x \in \Omega_1, \tag{64}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(D(x)\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(T(x)\frac{\mathrm{d}w(x)}{\mathrm{d}x}\right) - \omega^2 m(x)w(x) = 0, \quad \forall x \in \Omega_2 \tag{65}$$

and the conditions defined by Eqs. (45)–(51).

In this case the problem of finding a number $\bar{\lambda}$ and a function w such that

$$\begin{cases} w \in V, \quad w \neq 0, \\ B(w, v) - \bar{\lambda}(w, v) = 0, \quad \forall v \in V, \end{cases} \tag{66}$$

where $\bar{\lambda} = \rho(0)A(0)\omega^2$, $(w, v) = \int_\Omega h(x)w(x)v(x)\,dx$, is the eigenvalue problem of the bilinear form $B(w, v)$. The function $h(x)$ is given by $h(x) = h_1(x)h_2(x)$ where $\rho(x) = \rho(0)h_1(x)$ and $A(x) = A(0)h_2(x)$. If $B(w, v)$ is symmetric, continuous and V–elliptic, then it has a countable set of eigenvalues and are given by Necas [47] and Rektorys [48]:

$$\bar{\lambda}_1 = \min\left\{\frac{B(v, v)}{(v, v)}, \quad v \in V, \quad v \neq 0\right\}, \tag{67}$$

$$\bar{\lambda}_N = \min\left\{\frac{B(v, v)}{(v, v)} \quad v \in V, \quad v \neq 0, \quad (v, v_1) = \cdots = (v, v_N) = 0\right\}. \tag{68}$$

Let us introduce a new inner product in space V given by $((w, v)) = B(w, v)$, $\forall v \in V$. If the sequence $\{\varphi_i(x)\}_{i=1}^{\infty}$ is a base in the space V with the inner product $((w, v))$, the Ritz method leads to the equation

$$\begin{vmatrix} ((\varphi_1, \varphi_1)) - \bar{\lambda}(\varphi_1, \varphi_1) & \cdots & ((\varphi_1, \varphi_N)) - \bar{\lambda}(\varphi_1, \varphi_N) \\ & \vdots & \\ ((\varphi_N, \varphi_1)) - \bar{\lambda}(\varphi_N, \varphi_1) & \cdots & ((\varphi_N, \varphi_N)) - \bar{\lambda}(\varphi_N, \varphi_N) \end{vmatrix} = 0. \tag{69}$$

The approximate eigenvalues can be obtained from Eq. (69), when dealing with the dynamical behaviour of the beam considered above.

5. Natural frequencies and mode shapes

In order to check the accuracy of the algorithms developed, the frequency parameters were computed for a number of beams problems for which comparison values were available in the pertinent literature. Additionally, a great number of problems were solved and since the number of cases was extremely large, results were selected for the most significant cases. The analytical expressions obtained allow the adoption of different values for the following parameters:

(1) mass per unit length and flexural rigidity, of the ith span,
(2) rotational and translational restraint coefficients,
(3) axial forces T_1, T_c and T_2,
(4) distributed force $f(x)$,
(5) position of point c.

Using the well-known method of separation of variables, when the mass per unit length, the flexural rigidity and the axial force at the ith span are constant, we assume as solutions of Eqs. (16) and (17), respectively, the expressions

$$u_1(x, t) = \sum_{n=1}^{\infty} u_{1,n}(x) \cos \omega t \tag{70}$$

and

$$u_2(x, t) = \sum_{n=1}^{\infty} u_{2,n}(x) \cos \omega t, \tag{71}$$

where $u_{1,n}(x)$ and $u_{2,n}(x)$ are the corresponding nth modes of natural vibration.

Introducing the change of variable $\bar{x} = x/l$, into Eqs. (16)–(24) the functions $u_{1,n}(x)$ and $u_{2,n}(x)$ are given by

$$u_{1,n}(\bar{x}) = A_1 \cosh a_1\bar{x} + A_2 \sinh a_1\bar{x} + A_3 \cos b_1\bar{x} + A_4 \sin b_1\bar{x}, \tag{72}$$

$$u_{2,n}(\bar{x}) = A_5 \cosh a_2\bar{x} + A_6 \sinh a_2\bar{x} + A_7 \cos b_2\bar{x} + A_8 \sin b_2\bar{x} \tag{73}$$

and the following dimensionless parameters can be defined:

$$a_i = \sqrt{\frac{S_i}{2} + \Delta_i}, \quad b_i = \sqrt{-\frac{S_i}{2} + \Delta_i}, \quad \Delta_i = \sqrt{\frac{S_i^2}{4} + \lambda_i^4}, \quad i = 1, 2, \tag{74a–f}$$

$$S_i = \frac{T_i l^2}{D_i}, \quad \lambda_i^4 = \frac{\omega^2 m_i}{D_i} l^4, \quad i = 1, 2, \quad \lambda_2^4 = \frac{m_2}{m_1}\frac{D_1}{D_2}\lambda_1^4. \tag{75a–e}$$

Substituting Eqs. (72) and (73) into Eqs. (70) and (71) and then in the boundary conditions given by Eqs. (18), (19), (23) and (24) and transition conditions defined by Eqs. (26)–(29) we obtain a set of eight homogeneous equations in the constants A_i. Since the system is homogeneous for existence of a non-trivial solution the determinant of coefficients must be equal to zero. This procedure yields the frequency equation:

$$G(K_{ri}, K_{ti}, K_{rc}, K_{r12}, K_{tc}, S_i, \lambda_1, c) = 0, \tag{76}$$

where

$$K_{ri} = \frac{r_i l}{D_i}, \quad K_{ti} = \frac{t_i l^3}{D_i}, \quad i = 1, 2, \tag{77a,d}$$

$$K_{rc} = \frac{r_c l}{D_1}, \quad K_{r12} = \frac{r_{12} l}{D_1}, \quad K_{tc} = \frac{t_c l^3}{D_1}. \tag{78a–c}$$

The values of the frequency parameter $\lambda_1 = (\omega^2 m_1/D_1)^{1/4} l$, were obtained with the classical bisection method and rounded to six decimal digits. Table 4 depicts the first three exact values of the frequency parameter λ_1 of a uniform beam with a free internal hinge. Two different boundary conditions and values of S_1 and S_2 are considered. A comparison of values of the fundamental frequency with those of Ref. [36] given in plots, shows an excellent agreement from an engineering viewpoint. Table 5 depicts exact values of the fundamental frequency parameter $\lambda_{1,1}$ of a free–free stepped beam for different values of m_2/m_1 and D_2/D_1. The comparison with results of Ref. [49] shows an excellent agreement.

Table 4
First three exact values of the frequency parameter λ_1, of a uniform beam with and free internal hinge, two boundary conditions and two different values of S_1 and S_2

$S_1 = S_2$	c/l	C–C			C–SS		
		$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$	$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$
20	0.5	4.803415	8.289007	10.080855	4.332703	7.492574	9.573966
	0.4	4.884738	7.836494	11.003137	4.455115	7.012395	10.567871
	0.3	5.099945	7.439040	11.009806	4.605980	6.853667	10.111099
	0.2	5.218508	7.856133	10.321988	4.621299	7.367559	9.649816
	0.1	5.001689	8.225792	11.338858	4.425578	7.514569	10.625665
	0	4.642849	7.609618	10.625710	4.143643	6.961131	9.915604
50	0.5	5.562708	8.831742	10.881771	5.172028	8.197910	10.392662
	0.4	5.598529	8.549851	11.566439	5.217433	7.904708	11.116684
	0.3	5.692301	8.288043	11.570925	5.283389	7.786504	10.800847
	0.2	5.759859	8.542097	11.034230	5.310188	8.074734	10.447595
	0.1	5.636741	8.795938	11.810324	5.196631	8.197825	11.167929
	0	5.312890	8.249932	11.169513	4.930336	7.709381	10.537889

C: clamped, SS: simply supported.

Table 5
Exact fundamental frequency parameter $\lambda_{1,1}$ of a free–free stepped beam and different mass per unit length and flexural rigidity ratios ($m_2/m_1 = 0.5$, 0.75, $D_2/D_1 = 0.4$, 0.6, 0.8, 1)

D_2/D_1	$\lambda_{1,1}$	$m_2/m_1 = 0.5$	$m_2/m_1 = 0.75$
1.0	Present	5.11	4.89
	Ref. [49]	5.11	4.89
0.8	Present	4.98	4.76
	Ref. [49]	4.98	4.76
0.6	Present	4.80	4.57
	Ref. [49]	4.80	4.57
0.4	Present	4.51	4.27
	Ref. [49]	4.51	4.27

Table 6
Convergence of first three frequency parameters λ_1 of a uniform beam subjected to a variable axial force $T(\bar{x}) = T_1 + (T_2 - T_1)\bar{x}$, $\forall \bar{x} \in [0, 1]$, with $T_1 l^2/D = 0$ and $T_2 l^2/D - T_1 l^2/D = 4$

Boundary conditions	N	$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$
C–C				
	7	4.7869	7.9764	11.3296
	8	4.7869	7.9015	11.3296
	9	4.7869	7.9015	11.0427
	10	4.7869	7.9002	11.0427
	11	4.7869	7.9002	11.0326
	12	4.7869	7.9002	11.0326
	Ref. [50]	4.7869	7.9002	11.0326
SS–SS				
	7	3.2884	6.3739	9.5562
	8	3.2884	6.3611	9.5561
	9	3.2884	6.3611	9.4784
	10	3.2884	6.3611	9.4784
	11	3.2884	6.3611	9.4773
	12	3.2884	6.3611	9.4773
	Ref. [50]	3.2884	6.3611	9.4773
F–F				
	7	2.1771	4.9449	8.0840
	8	2.1771	4.9449	7.9639
	9	2.1771	4.9449	7.9638
	10	2.1771	4.9449	7.9629
	11	2.1771	4.9449	7.9629
	12	2.1771	4.9449	7.9629
	Ref. [50]	2.1771	4.9449	7.9629
F–SS				
	7	1.5634	4.0703	7.1804
	8	1.5634	4.0703	7.1428
	9	1.5634	4.0703	7.1425
	10	1.5634	4.0703	7.1423
	11	1.5634	4.0703	7.1423
	12	1.5634	4.0703	7.1423
	Ref. [50]	1.5634	4.0703	7.1423

Table 7
First three approximate values of the frequency parameter λ_1 of a uniform and beam subjected to a variable axial force $T(\bar{x}) = T_1 + (T_2 - T_1)\bar{x}$, $\forall \bar{x} \in [0, 1]$ with $T_1 l^2/D = 10$ and $T_2 l^2/D - T_1 l^2/D = 100$

Boundary conditions	$\lambda_{1,1}$		$\lambda_{1,2}$		$\lambda_{1,3}$	
	Present	Ref. [50]	Present	Ref. [50]	Present	Ref. [50]
C–C	5.8768	5.8768	8.9720	8.9720	11.9595	11.9595
C–S	5.5709	5.5709	8.5122	8.5122	11.3932	11.3932
C–SL	3.5912	3.5912	7.0372	7.0372	9.9341	9.9341
C–F	3.5876	3.5876	6.9736	6.9736	9.7435	9.7435
SS–C	5.2505	5.2505	8.2814	8.2814	11.2405	11.2405
SS–SS	5.0032	5.0032	7.8617	7.8617	10.6990	10.6990
SS–SL	3.0737	3.0737	6.4101	6.4101	9.2566	9.2566
SS–F	3.0722	3.0722	6.3678	6.3678	9.1013	9.1013
SL–C	3.8567	3.8567	6.8370	6.8370	9.7732	9.7732
SL–SS	3.7150	3.7150	6.5037	6.5037	9.2877	9.2877
SL–SL	5.0483	5.0483	7.8632	7.8632	10.6986	10.6986
SL–F	5.0346	5.0346	7.7720	7.7720	10.4718	10.4718
F–C	3.7806	3.7806	6.3957	6.3957	9.1596	9.1596
F–SS	3.6551	3.6551	6.1204	6.1204	8.7159	8.7159
F–SL	4.8344	4.8344	7.3614	7.3614	10.0767	10.0767
F–F	4.8247	4.8247	7.2942	7.2942	9.8855	9.8855

Results of a convergence study of the first three values of the frequency parameter λ_1 are presented in Table 6. A uniform beam subjected to a variable axial force $T(\bar{x}) = T_1 + (T_2 - T_1)\bar{x}$, $\forall \bar{x} \in [0, 1]$, with $T_1 l^2/D = 0$ and $T_2 l^2/D - T_1 l^2/D = 4$, and different boundary conditions is considered. These values have been calculated by using the Ritz method adopting the system $\{1, \bar{x}, \bar{x}^2, \ldots, \bar{x}^N, \ldots\}$ as a base. The eigenvalues have been determined from Eq. (69). It is well known that the Ritz method gives upper bounds eigenvalues. The convergence of the mentioned eigenvalues is studied by gradually increasing the number of the elements of the base. It can be seen that the use of $N = 12$ is sufficient to reach stable convergence in all cases.

Table 7 depicts the first three approximate values of the frequency parameter λ_1 of a uniform beam subjected to a variable axial force $T(\bar{x}) = T_1 + (T_2 - T_1)\bar{x}$, $\forall \bar{x} \in [0, 1]$, with $T_1 l^2/D = 10$ and $T_2 l^2/D - T_1 l^2/D = 100$. These values have been calculated by using the Ritz method with $N = 12$. The comparison with results of Ref. [50] shows a very close agreement.

Table 8 depicts the first three exact values of the frequency parameter λ_1 of a uniform beam with a free internal hinge and different boundary conditions. The mode shapes which correspond to a hinge located at $c/l = 0.5$ are also presented.

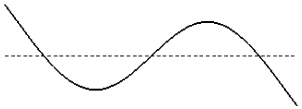
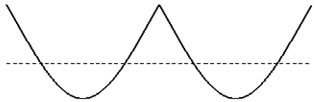
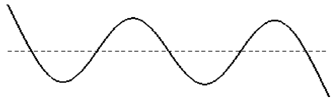
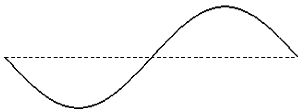
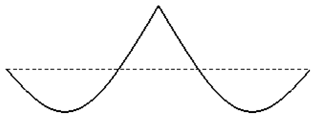
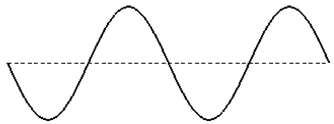
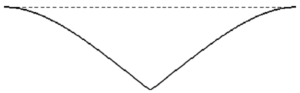
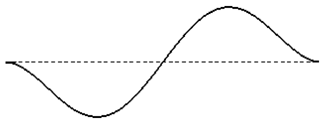
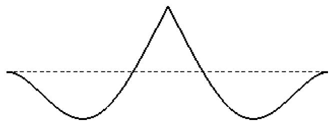
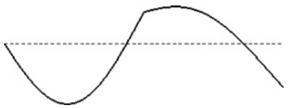
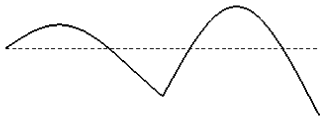
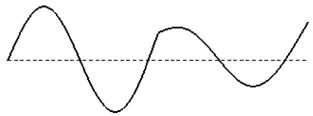
In Fig. 2 the exact fundamental frequency parameter $\lambda_{1,1}$ is plotted against the restraint parameters K_{r2} and K_{t2}. The beam is clamped at $\bar{x} = 0$ and elastically restrained at $\bar{x} = 1$. The free internal hinge ($K_{r12} = K_{rc} = K_{tc} = 0$) is located at $c/l = 0.5$. It can be observed that major increase of frequency occur when the elastic restrain values are in the interval [0.01, 10].

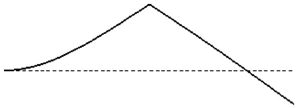
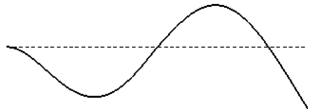
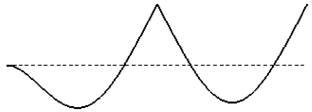
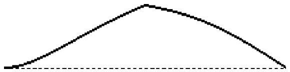
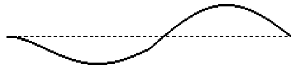
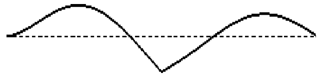
Table 9 depicts exact values of the fundamental frequency parameter $\lambda_{1,1}$, of a uniform beam clamped at $\bar{x} = 0$ with an intermediate point elastically restrained against rotation and translation, with an elastically restrained internal hinge and free at $\bar{x} = 1$. When $K_{r12} \to \infty$ the values obtained agree with those of Ref. [33].

6. Conclusions

A simple, computationally efficient and accurate approach has been developed for the determination of natural frequencies and modal shapes of free vibration of a non-homogeneous tapered beam subjected to general axial forces, with arbitrarily located internal hinge and elastics supports, and ends elastically restrained against rotation and translation. Hamilton's principle has been rigorously applied to obtain the differential equations, boundary conditions, and particularly the transitions conditions. The algorithm is very general and it is attractive regarding its versatility in handling any boundary conditions and any transition conditions,

Table 8
First three exact values of the frequency parameter λ_1 of a uniform beam with a free internal hinge and different boundary conditions (the mode shapes which correspond to a hinge located at $c/l = 0.5$ are also presented)

Boundary conditions	c/l	$\lambda_{1,1}$	Mode shape	$\lambda_{1,2}$	Mode shape	$\lambda_{1,3}$	Mode shape
F–F	0.5	7.853205		9.460081		14.137165	
	0.4	7.149421		10.610666		12.733213	
	0.3	6.301510		10.528024		13.933756	
	0.2	5.639917		9.421523		13.235406	
	0.1	5.129561		8.532251		11.964969	
	0	4.730041		7.853205		10.995608	
SS–SS	0.5	6.283185		7.853205		12.566371	
	0.4	5.782606		8.767856		11.312915	
	0.3	5.131786		9.276929		11.780378	
	0.2	4.618322		8.391546		12.161731	
	0.1	4.226369		7.631299		11.050510	
	0	3.141593		6.283185		9.424778	
C–C	0.5	3.750208		7.853205		9.388182	
	0.4	3.953408		7.180653		10.560987	
	0.3	4.483116		6.584228		10.573598	
	0.2	4.706597		7.225821		9.737986	
	0.1	4.348263		7.757249		10.987577	
	0	3.926602		7.068583		10.210176	
SS–F	1	3.926602		7.068583		10.210176	
	0.9	4.256359		7.676480		11.106217	
	0.8	4.673940		8.469451		12.283192	
	0.7	5.214137		9.478488		13.342975	
	0.6	5.922674		10.167962		11.798847	
	0.5	6.786463		8.926649		13.090828	
	0.4	6.920416		9.052879		12.468217	
	0.3	6.205473		10.198446		12.124760	
	0.2	5.577540		9.336974		13.089156	
	0.1	5.094911		8.484132		11.907033	
	0	3.926602		7.068583		10.210176	
C–F	1	1.875104		4.694091		7.854757	
	0.9	2.029130		5.090997		8.533888	
	0.8	2.216011		5.598924		9.423177	
	0.7	2.448374		6.258639		10.529426	
	0.6	2.746150		7.108354		10.608056	
	0.5	3.141593		7.853205		9.424778	

	0.4	3.683036		7.222925		10.563596	
	0.3	4.373695		6.623242		10.572130	
	0.2	4.682586		7.235941		9.736372	
	0.1	4.346531		7.757735		10.987489	
	0	3.926602		7.068583		10.210176	
C−SS	1	3.926602		7.068583		10.210176	
	0.9	2.012461		5.056885		8.485728	
	0.8	2.180143		5.538082		9.338492	
	0.7	2.389706		6.166247		10.198789	
	0.6	2.658637		6.894103		9.042906	
	0.5	3.011831		6.826201		8.874549	
	0.4	3.467002		6.068954		10.149537	
	0.3	3.885351		5.835217		9.544539	
	0.2	3.823514		6.740460		8.957675	
	0.1	3.482771		6.915324		10.209583	
	0	3.141593		6.283185		9.424778	

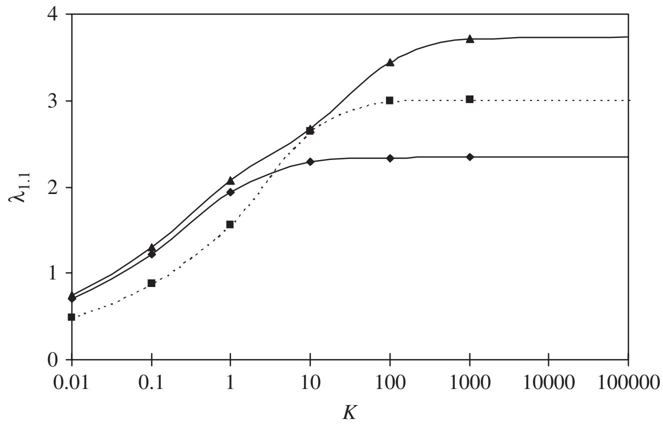


Fig. 2. Variation of fundamental frequency parameter $\lambda_{1,1}$ with rotational and translational restraint parameters K_{r2} and K_{t2} of a clamped-elastically restrained beam with a free internal hinge ($K_{12} = K_{rc} = K_{tc} = 0$) located at $c/l = 0.5$. (—◆—) $K_{t2} = 0$, $K_{r2} = K$. (- - -■- - -) $K_{t2} = K$, $K_{r2} = 0$. (—▲—) $K_{t2} = K$, $K_{r2} = K$.

Table 9
Exact values of the fundamental frequency parameter $\lambda_{1,1}$, of a uniform cantilever beam with an intermediate point elastically restrained against rotation and traslation ($K_{tc} = K_{rc}$) and with an elastically restrained internal hinge located at two different points, $c/l = 0.4$, 0.6

c	$K_{tc} = K_{rc}$	∞		K_{r12}				
				1000	100	10	1	0
		Present	Ref. [33]					
0.6	0	1.87510	1.87510	1.87500	1.87411	1.86527	1.78378	2.74615
	1	2.09119	2.09119	2.09102	2.08951	2.07456	1.94204	2.98916
	10	2.74618	2.74618	2.74558	2.74017	2.68723	2.29297	3.58787
	100	3.67938	3.67938	3.67670	3.65271	3.43269	2.49446	4.62004
	1000	4.53275	4.53275	4.52282	4.43794	3.86571	2.54981	6.84224
	10,000	4.67264	4.67264	4.66114	4.56340	3.92741	2.55730	7.79768
0.4	0	1.87510	–	1.87471	1.87114	1.83709	1.60366	3.68304
	1	2.01557	–	2.01499	2.00986	1.96170	1.66141	3.90061
	10	2.50929	–	2.50755	2.49212	2.35917	1.79295	4.54802
	100	2.95500	–	2.95100	2.91623	2.65068	1.85214	5.26730
	1000	3.09608	–	3.09110	3.04789	2.73066	1.86491	6.27871
	10,000	3.12195	–	3.11679	3.07205	2.74529	1.86719	6.51932

including ends and an intermediate point elastically restrained against rotation and translation. Besides, it allows to take into account a great variety of complicating effects such us: thickness variation, different types of axial forces and an arbitrarily located internal hinge with a rotational restraint. Different mass per unit length and flexural rigidity in each span can be considered. When the mass per unit length or the flexural rigidity or the axial force are variable at the ith span the Ritz method has been used. Close agreement with results presented by previous investigators is demonstrated for several examples. New results are presented for several beams with internal hinge and elastic restraints. These results may provide useful information for structural designers and engineers.

It has been demonstrated that the boundary and the eigenvalue problem which, respectively, describe the statical and dynamical behaviour of the mechanical system analysed, do not have classical solutions. The problem of existence and uniqueness of the weak solutions of the corresponding boundary value problem and eigenvalue problem, has been treated. The use of the weak solution theory enables a substantial generalization of assumptions concerning the coefficients smoothness of the corresponding differential equations and the continuity of the load $q(x)$ in the static case. Consequently problems involving non-uniform beams such us stepped beams, discontinuous loads, intermediate supports, etc., can be considered.

Acknowledgements

The authors are grateful to Dr. Carlos P. Filipich for his valuable comments and suggestions. The present investigation has been sponsored by CONICET Research and Development Program (PIP 5340).

Appendix A. Determination of Eq. (9) and Table A1

Let us consider the three integrals over Ω_i involved in Eq. (8). Since $u(\bullet, t), v(\bullet, t) \in C^2[t_a, t_b]$, we can apply the integration by parts method with respect to t and if we apply the conditions $v(x, t_a) = v(x, t_b) = 0$, $\forall x \in [0, l]$ imposed in Eq. (6), we obtain

$$\int_{t_a}^{t_b} \int_{\Omega_i} m(x) \frac{\partial u(x,t)}{\partial t} \frac{\partial v(x,t)}{\partial t} \, \mathrm{d}x \, \mathrm{d}t = -\int_{t_a}^{t_b} \int_{\Omega_i} m(x) \frac{\partial^2 u(x,t)}{\partial t^2} v(x,t) \, \mathrm{d}x \, \mathrm{d}t, \quad i = 1, 2. \tag{A.1}$$

Since $u(x, \bullet)\big|_{[0,c]}, v(x, \bullet)\big|_{[0,c]} \in C^4[0, c]$ and $u(x, \bullet)\big|_{[c,l]}, v(x, \bullet)\big|_{[c,l]} \in C^4[c, l]$, it is possible to apply twice the integration by parts method with respect to x in the integral where $D(x)$ is involved, to obtain

$$\begin{aligned}
\int_{t_a}^{t_b} \int_{\Omega_i} D(x) \frac{\partial^2 u(x,t)}{\partial x^2} \frac{\partial^2 v(x,t)}{\partial x^2} \mathrm{d}x \, \mathrm{d}t &= \int_{t_a}^{t_b} \int_{\Omega_i} \frac{\partial^2}{\partial x^2} \left(D(x) \frac{\partial^2 u(x,t)}{\partial x^2} \right) v(x,t) \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_{t_a}^{t_b} \left(D(b_{i+1}) \frac{\partial^2 u(b_{i+1},t)}{\partial x^2} \frac{\partial v(b_{i+1},t)}{\partial x} - D(d_i) \frac{\partial^2 u(d_i,t)}{\partial x^2} \frac{\partial v(d_i,t)}{\partial x} \right. \\
&\left. - \frac{\partial}{\partial x} \left(D(b_{i+1}) \frac{\partial^2 u(b_{i+1},t)}{\partial x^2} \right) v(b_{i+1},t) + \frac{\partial}{\partial x} \left(D(d_i) \frac{\partial^2 u(d_i,t)}{\partial x^2} \right) v(d_i,t) \right) \mathrm{d}t.
\end{aligned} \tag{A.2}$$

Finally, integrating by parts with respect to x in the integral where $T(x)$ is involved, we have

$$\begin{aligned}
\int_{t_a}^{t_b} \int_{\Omega_i} T(x) \frac{\partial u(x,t)}{\partial x} \frac{\partial v(x,t)}{\partial x} \, \mathrm{d}x \, \mathrm{d}t &= -\int_{t_a}^{t_b} \int_{\Omega_i} \frac{\partial}{\partial x} \left(T(x) \frac{\partial u(x,t)}{\partial x} \right) v(x,t) \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_{t_a}^{t_b} \left(T(b_{i+1}) \frac{\partial u(b_{i+1},t)}{\partial x} v(b_{i+1},t) - T(d_i) \frac{\partial u(d_i,t)}{\partial x} v(d_i,t) \right) \mathrm{d}t.
\end{aligned} \tag{A.3}$$

Replacing (A.1)–(A.3) into Eq. (8) we obtain Eq. (9), i.e.:

$$\begin{aligned}
\delta F(u; v) = & -\int_{t_a}^{t_b} \left\{ \sum_{i=1}^{2} \int_{\Omega_i} \left[m(x) \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left(D(x) \frac{\partial^2 u(x,t)}{\partial x^2} \right) \right. \right. \\
& \left. \left. - \frac{\partial}{\partial x} \left(T(x) \frac{\partial u(x,t)}{\partial x} \right) \right] v(x,t) \, \mathrm{d}x \right\} \mathrm{d}t + \int_{t_a}^{t_b} \sum_{i=1}^{2} \left(P_i \frac{\partial v(A_i,t)}{\partial x} + Q_i v(A_i,t) \right. \\
& \left. + R_i \frac{\partial v(B_i,t)}{\partial x} + Z_i v(c,t) \right) \mathrm{d}t,
\end{aligned} \tag{A.4}$$

where

$$\begin{aligned}
P_i &= -r_i \frac{\partial u(A_i,t)}{\partial x} + (-1)^{i+1} D(A_i) \frac{\partial^2 u(A_i,t)}{\partial x^2}, \\
Q_i &= -t_i u(A_i,t) + (-1)^i \left[\frac{\partial}{\partial x} \left(D(A_i) \frac{\partial^2 u(A_i,t)}{\partial x^2} \right) - T(A_i) \frac{\partial u(A_i,t)}{\partial x} \right], \\
R_i &= (-1)^{i-1} \left(r_{12} \left(\frac{\partial u(c^+,t)}{\partial x} - \frac{\partial u(c^-,t)}{\partial x} \right) - D(B_i) \frac{\partial^2 u(B_i,t)}{\partial x^2} \right) - C_i \frac{\partial u(B_i,t)}{\partial x}, \\
Z_i &= -\frac{1}{2} t_c u(c,t) + (-1)^{i-1} \left(\frac{\partial}{\partial x} \left(D(B_i) \frac{\partial^2 u(B_i,t)}{\partial x^2} \right) - T(B_i) \frac{\partial u(B_i,t)}{\partial x} \right), \\
A_1 &= 0^+, \quad A_2 = l^-, \quad B_1 = c^-, \quad B_2 = c^+, \quad C_1 = r_c, \quad C_2 = 0.
\end{aligned}$$

Table A1
Transition conditions (the condition $u(c^-, t) = u(c^+, t)$, corresponds to all the cases)

Case	t_c	r_c	r_{12}	Transition conditions
1	0	0	∞	$\frac{\partial u(c^-,t)}{\partial x} = \frac{\partial u(c^+,t)}{\partial x}, \quad D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2} = D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2},$ $\frac{\partial u(c^-,t)}{\partial x}\left(T(c^-) - T(c^+)\right) = \frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}\right).$
2	0	0	0	$D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2} = D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2} = 0,$ $T(c^-)\frac{\partial u(c^-,t)}{\partial x} - T(c^+)\frac{\partial u(c^+,t)}{\partial x} = \frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}\right).$
3	0	∞	0	$\frac{\partial u(c^-,t)}{\partial x} = 0, \quad D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2} = 0,$ $T(c^-)\frac{\partial u(c^-,t)}{\partial x} - T(c^+)\frac{\partial u(c^+,t)}{\partial x} = \frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}\right).$
4	∞	0	0	$u(c,t) = 0, \quad D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2} = D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2} = 0.$
5	∞	0	∞	$u(c,t) = 0, \quad \frac{\partial u(c^-,t)}{\partial x} = \frac{\partial u(c^+,t)}{\partial x}, \quad D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2} = D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}.$
6	0	$0 < r_c < \infty$	0	$r_c\frac{\partial u(c^-,t)}{\partial x} = -D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}, \quad D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2} = 0,$ $T(c^-)\frac{\partial u(c^-,t)}{\partial x} - T(c^+)\frac{\partial u(c^+,t)}{\partial x} = \frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}\right).$
7	$0 < t_c < \infty$	0	0	$D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2} = D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2} = 0,$ $t_c u(c,t) = \frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}\right) - T(c^-)\frac{\partial u(c^-,t)}{\partial x} + T(c^+)\frac{\partial u(c^+,t)}{\partial x}.$
8	∞	∞	$0 < r_{12} < \infty$	$u(c,t) = 0, \quad \frac{\partial u(c^-,t)}{\partial x} = 0, \quad r_{12}\frac{\partial u(c^+,t)}{\partial x} = D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}.$

9	0	0	$0<r_{12}<\infty$	$D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}=D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2},\quad r_{12}\left(\frac{\partial u(c^+,t)}{\partial x}-\frac{\partial u(c^-,t)}{\partial x}\right)=D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2},$ $T(c^-)\frac{\partial u(c^-,t)}{\partial x}-T(c^+)\frac{\partial u(c^+,t)}{\partial x}=\frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}\right).$
10	0	∞	$0<r_{12}<\infty$	$\frac{\partial u(c^-,t)}{\partial x}=0,\quad r_{12}\frac{\partial u(c^+,t)}{\partial x}=D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2},$ $T(c^-)\frac{\partial u(c^-,t)}{\partial x}-T(c^+)\frac{\partial u(c^+,t)}{\partial x}=\frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}\right).$
11	∞	0	$0<r_{12}<\infty$	$u(c,t)=0,$ $D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}=D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2},\quad r_{12}\left(\frac{\partial u(c^+,t)}{\partial x}-\frac{\partial u(c^-,t)}{\partial x}\right)=D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}.$
12	∞	$0<r_c<\infty$	0	$u(c,t)=0,$ $D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}=0,\quad r_c\frac{\partial u(c^-,t)}{\partial x}=-D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}.$
13	$0<t_c<\infty$	$0<r_c<\infty$	∞	$\frac{\partial u(c^-,t)}{\partial x}=\frac{\partial u(c^+,t)}{\partial x},\quad r_c\frac{\partial u(c^-,t)}{\partial x}=D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}-D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2},$ $t_c u(c,t)=\frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}\right)-T(c^-)\frac{\partial u(c^-,t)}{\partial x}+T(c^+)\frac{\partial u(c^+,t)}{\partial x}.$
14	$0<t_c<\infty$	0	$0<r_{12}<\infty$	$D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}=D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2},\quad r_{12}\left(\frac{\partial u(c^+,t)}{\partial x}-\frac{\partial u(c^-,t)}{\partial x}\right)=D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2},$ $t_c u(c,t)=\frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}\right)-T(c^-)\frac{\partial u(c^-,t)}{\partial x}+T(c^+)\frac{\partial u(c^+,t)}{\partial x}.$
15	$0<t_c<\infty$	$0<r_c<\infty$	$0<r_{12}<\infty$	$r_{12}\left(\frac{\partial u(c^+,t)}{\partial x}-\frac{\partial u(c^-,t)}{\partial x}\right)=D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2},\quad r_c\frac{\partial u(c^-,t)}{\partial x}=D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}-D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2},$ $t_c u(c,t)=\frac{\partial}{\partial x}\left(D(c^-)\frac{\partial^2 u(c^-,t)}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(D(c^+)\frac{\partial^2 u(c^+,t)}{\partial x^2}\right)-T(c^-)\frac{\partial u(c^-,t)}{\partial x}+T(c^+)\frac{\partial u(c^+,t)}{\partial x}.$

References

[1] I. Guelfand, S. Fomin, *Calculus of Variations*, Prentice-Hall, Englewood Cliffs, NJ, 1963.
[2] J.L. Troutman, *Variational Calculus and Optimal Control*, Springer, New York, 1996.
[3] M. Giaquinta, S. Hildebrandt, *Calculus of Variations I*, Springer, Berlín, 1996.
[4] C. Dym, I. Shames, *Solid Mechanics: A Variational Approach*, Mc Graw Hill Book Company, New York, 1973.
[5] L. Kantorovich, V. Krylov, *Approximate Methods of Higher Analysis*, Interscience Publishers, New York, 1964.
[6] F. Hildebrand, *Methods of Applied Mathematics*, Prentice-Hall, Englewood Cliffs, NJ, 1965.
[7] R. Weinstock, *Calculus of Variations with Applications to Physics and Engineering*, Dover Publications, New York, 1974.
[8] L. Elsgoltz, *Calculus of Variations*, Addison-Wesley, Publishing Co., Reading, MA, 1962.
[9] R.D. Blevins, *Formulas for Natural Frequency and Mode Shape*, Van Nostrand Reinhold, New York, 1979.
[10] I.A. Karnovsky, O.I. Lebed, *Free Vibrations of Beams and Frames*, Mc Graw-Hill, New York, 2004.
[11] W. Lee, Vibration frequencies for a uniform beam with one end spring hinged and carrying a mass at the other free end, *Journal of Applied Mechanics* 40 (1973) 813–815.
[12] A. Grant, Vibration frequencies for a uniform beam with one end elastically supported and carrying a mass at the other end, *Journal of Applied Mechanics* 42 (1975) 878–880.
[13] C. Hibbeler, Free vibrations of a beam supported with unsymmetrical spring-hinges, *Journal of Applied Mechanics* 42 (1975) 501–502.
[14] R. Maurizi, R. Rossi R, J. Reyes, Vibration frequencies for a uniform beam with one end spring hinged and subjected to a translational restraint at the other end, *Journal of Sound and Vibration* 48 (1976) 565–568.
[15] P. Goel, Free vibrations of a beam–mass system with elastically restrained ends, *Journal of Sound and Vibration* 47 (1976) 9–14.
[16] R.O. Grossi, P.A.A. Laura, Further results on a vibrating beam with a mass and spring at the end subjected to an axial force, *Journal of Sound and Vibration* 84 (1982) 593–594.
[17] P.A.A. Laura, R.O. Grossi, S. Alvarez, Transverse vibrations of a beam elastically restrained at one end and with a mass and spring at the other subjected to an axial force, *Nuclear Engineering and Design* 74 (1982) 299–302.
[18] C.K. Rao, S. Mirza, Note on vibrations of generally restrained beams, *Journal of Sound and Vibration* 130 (1989) 453–465.
[19] H. Mabie, C.B. Rogers, Transverse vibrations of tapered cantilever beams with end support, *Journal of the Acoustical Society of America* 44 (1968) 1739–1741.
[20] H. Mabie, C.B. Rogers, Transverse vibrations of double-tapered cantilever beams, *Journal of the Acoustical Society of America* 57 (1972) 1771–1774.
[21] H. Mabie, C.B. Rogers, Transverse vibrations of double-tapered cantilever beams with end support and with end mass, *Journal of the Acoustical Society of America* 55 (1974) 986–991.
[22] P. Goel, Transverse vibrations of tapered beams, *Journal of Sound and Vibration* 47 (1976) 1–7.
[23] H. Cortinez, P.A.A. Laura, Vibration and buckling of a non-uniform beam elastically restrained against rotation at one end and with concentrated mass at the other, *Journal of Sound and Vibration* 99 (1985) 144–148.
[24] P.A. Laura, R.H. Gutierrez, Vibration of a elastically restrained cantilever beam of varying cross section with tip mass of finite length, *Journal of Sound and Vibration* 108 (1986) 123–131.
[25] R.O. Grossi, R.B. Bhat, A note on vibrating tapered beams, *Journal of Sound and Vibration* 47 (1991) 174–178.
[26] R.O. Grossi, A. Aranda, R.B. Bhat, Vibration of tapered beams with one end spring hinged and the other end with tip mass, *Journal of Sound and Vibration* 160 (1993) 175–178.
[27] L. Nallim, R.O. Grossi, A general algorithm for the study of the dynamical behaviour of beams, *Applied Acoustics* 57 (1999) 345–356.
[28] A. Rutemberg, Vibration frequencies for a uniform cantilever with a rotational constraint at a point, *Journal of Applied Mechanics* 45 (1978) 123–422.
[29] J.H. Lau, Vibration frequencies and mode shapes for a constrained cantilever, *Journal of Applied Mechanics* 57 (1984) 182–187.
[30] C.K. Rao, Frequency analysis of clamped–clamped uniform beams with intermediate elastic support, *Journal of Sound and Vibration* 133 (1989) 502–509.
[31] M.A. de Rosa, P.M. Bellés, M.J. Maurizi, Free vibrations of stepped beams with intermediate elastic supports, *Journal of Sound and Vibration* 181 (1995) 905–910.
[32] B. Arenas, R.O. Grossi, Vibration frequencies for a beam with a rotational restraint in an adjustable position, *Applied Acoustics* 57 (1999) 197–202.
[33] R.O. Grossi, C. Albarracín, Eigenfrequencies of generally restrained beams, *Journal of Applied Mathematics* 10 (2003) 503–516.
[34] C.Y. Wang, Minimum stiffness of an internal elastic support to maximize the fundamental frequency of a vibrating beam, *Journal of Sound and Vibration* 259 (2003) 229–232.
[35] M.S. Ewing, S. Mirsafian, Forced vibrations of two beams joined with a non-linear rotational joint: clamped and simply supported end conditions, *Journal of Sound and Vibration* 193 (1996) 483–496.
[36] C.Y. Wang, C.M. Wang, Vibrations of a beam with an internal hinge, *International Journal of Structural Stability and Dynamics* 1 (2001) 163–167.
[37] T.P. Chang, G.L. Lin, E. Chang, Vibrations analysis of a beam with an internal hinge subjected to a random moving oscillator, *International Journal of Solid and Structures* 43 (2006) 6398–6412.
[38] C. Baiocchi, A. Capelo, *Variational and Quasivariational Inequalities*, Vol. 1, Pitagora Editrice, Bologna, 1978 (in Italian).
[39] H. Brezis, *Functional Analysis*, Masson, Paris, 1983 (in French).
[40] J. Reddy, *Applied Functional Analysis and Variational Methods in Engineering*, McGraw-Hill, New York, 1986.

[41] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, Vol. IIA, Springer, New York, 1990.
[42] E. Zeidler, *Applied Functional Analysis: Main Principles and their Applications*, Vol. 109, Springer, New York, 1995.
[43] E. Zeidler, *Applied Functional Analysis: Applications to Mathematical Physics*, Vol. 108, Springer, New York, 1995.
[44] J.T. Oden, L.F. Demkowicz, *Applied Functional Analysis*, CRC Press, Boca Ratón, FL, 1996.
[45] P.A. Raviart, J.M. Thomas, *Introduction to the Numerical Analysis of Partial Differential Equations*, Dunod, Paris, 1998 (in French).
[46] L.C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, RI, 1998.
[47] J. Necas, *The Direct Methods in the Theory of Elliptic Equations*, Prage, Academia, 1967 (in French).
[48] K. Rektorys, *Variational Methods in Mathematics, Science and Engineering*, Reidel Co., Dordrecht, 1980.
[49] K. Kanaka Raju, G.V. Rao, Free vibrations of a free–free partially stressed stepped beam with follower forces, *Journal of Sound and Vibration* 178 (1994) 429–432.
[50] S. Naguleswaran, Transverse vibrations of a uniform Euler–Bernoulli beam under linearly varying axial force, *Journal of Sound and Vibration* 275 (2004) 47–57.